Asymptotic Expansions and Influence Coefficients for Edge-Loaded Conical Shells

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The equations of linear elasticity for rotationally symmetric deformations are expanded using a small parameter related to the thickness to radius of curvature ratio of the shell to obtain the classical thin shell equations of conical shells as a first approximation. These classical equations with variable coefficients permit further asymptotic expansions in the cases of steep as well as shallow cones, yielding systems of equations with constant coefficients. Solutions of these equations are used to compute the influence coefficients relating edge loads and edge displacements.

Introduction

In the case of circular cylindrical shells, asymptotic expansions of the equations of elasticity resulting in a sequence of systems of equations, the first of which is the classical theory, have been obtained by Johnson and Reissner [1]. Further results for cylindrical shells using these expansions have been reported by Reissner and coworkers [2–5]. A study of conical shells along the lines of [1] has not been carried out so far. In the present paper first we show that such a derivation is indeed possible. We list two sets of differential equations, the first being the classical conical shell theory and the second representing the effects due to finite thickness. We also derive a characteristic length for the edge bending effect, which varies from the entire domain in the case of a shallow conical shell

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to the characteristic length associated with the cylindrical shell for high conicity. The classical theory of conical shells is embodied in a system of differential equations which can be reduced to a single fourth order differential equation with variable coefficients. The solutions of these equations are in terms of Kelvin functions [6], which are related to Bessel functions. Here we show that the introduction of a second small parameter related to the conicity of the shell can be used to perturb the classical equations to obtain an asymptotic sequence, the first of which represents a cylindrical shell and the effects of conicity appearing in the higher order systems. This small parameter is in fact the ratio of the characteristic length to the length of the shell. Interestingly, a one term correction gives sufficient accuracy even for apparently shallow cones with semiapex angles as high as 70°.

The results for steep conical shells are supplemented with asymptotic results for shallow shells using a second small parameter. Effectively, we consider shallow shells as obtained by perturbing a circular plate. Finally, for the sake of comparison the exact influence coefficients are computed using the Kelvin functions. Numerical results are given in tabular form.

Formulation of the problem

We consider the equations of linear elasticity for rotationally symmetric deformations in polar coordinates (r, θ, z) in the form

$$r\sigma_{z,z} + (r\sigma_{zr})_{,r} = 0, \qquad r\sigma_{zr,z} + (r\sigma_r)_{,r} - \sigma_\theta = 0, \tag{1}$$

$$\epsilon_z = u_{z,z}, \qquad \epsilon_\theta = u_r/r, \qquad \epsilon_r = u_{r,r}, \qquad 2\epsilon_{zr} = u_{z,r} + u_{r,z}. \tag{2}$$

Introducing the coordinate transformation

$$x = z\cos\alpha + r\sin\alpha, \qquad y = r\cos\alpha - z\sin\alpha, \tag{3}$$

we define a conical shell with semiapex angle α, (slant) length l, and semithickness c as the domain: $0 < x < l$, $-c < y < c$, $0 < \theta \leq 2\pi$.

Denoting the displacement components by (u_x, u_y, u_θ) and stress components by $(\sigma_x, \sigma_y, \sigma_\theta, \sigma_{xy})$, the equations (1) and (2) can be written as

$$\begin{aligned} &[(x + y\cot\alpha)\sigma_x]_{,x} + (x + y\cot\alpha)\sigma_{x,y} - \sigma_\theta = 0 \\ &[(x + y\cot\alpha)\sigma_{xy}]_{,x} + (x + y\cot\alpha)\sigma_{r,r} - \sigma_\theta\cot\alpha = 0 \end{aligned} \tag{4}$$

and

$$\begin{aligned} &\epsilon_x = u_{x,x}, \qquad \epsilon_\theta = \frac{u_x + u_y\cot\alpha}{x + y\cot\alpha} \\ &\epsilon_y = u_{y,y}, \qquad 2\epsilon_{xy} = u_{x,y} + u_{y,x}. \end{aligned} \tag{5}$$

These equations are supplemented by the constitutive relations

$$E\epsilon_x = \sigma_x - \nu(\sigma_\theta + \sigma_y), \qquad E\epsilon_\theta = \sigma_\theta - \nu(\sigma_x + \sigma_y),$$
$$E\epsilon_y = \sigma_y - \nu(\sigma_x + \sigma_\theta), \qquad E\epsilon_{xy} = (1+\nu)\sigma_{xy}. \tag{6}$$

We assume the inner and outer surfaces of the shell are free of tractions and the edge $x = l$ is subjected to prescribed normal stress $\bar{\sigma}_x(y)$ and shear stress $\bar{\sigma}_{xy}(y)$ in such a way that overall equilibrium is maintained. These conditions can be written as

$$0 < x < l, \quad y = \pm c: \qquad \sigma_y = 0, \quad \sigma_{xy} = 0; \tag{7}$$

$$x = l, \quad -c < y < c: \qquad \sigma_x = \bar{\sigma}_x(y), \quad \sigma_{xy} = \bar{\sigma}_{xy}(y); \tag{8}$$

$$\int_{-c}^{c} \bar{\sigma}_x\left(1 + \frac{y}{R_0}\right) dy = \tan\alpha \int_{-c}^{c} \bar{\sigma}_{xy}\left(1 + \frac{y}{R_0}\right) dy, \tag{9}$$

where $R_0 = l\tan\alpha$ is the smaller radius of principal curvature at the edge of the shell.

At any point x on the midsurface of the shell we define stress resultants and moments

$$N_x = \int_{-c}^{c} \sigma_x\left(1 + \frac{y}{R}\right) dy, \qquad N_\theta = \int_{-c}^{c} \sigma_\theta \, dy, \tag{10}$$

$$M_x = \int_{-c}^{c} \sigma_x\left(1 + \frac{y}{R}\right) y \, dy, \tag{11}$$

$$M_\theta = \int_{-c}^{c} \sigma_\theta y \, dy \tag{12}$$

$$Q_x = \int_{-c}^{c} \sigma_{xy}\left(1 + \frac{y}{R}\right) dy \tag{13}$$

where $R = x\tan\alpha$.

Considering the linearity of the problem, it is convenient to divide the boundary conditions (8) and (9) into two sets:

$$M_x(0) = M_0, \qquad N_x(0) = 0, \qquad Q_x(0) = 0 \tag{14}$$

and

$$Q_x(0) = Q_0, \qquad N_x(0) = Q_0 \tan\alpha, \qquad M_x(0) = 0. \tag{15}$$

If the prescribed functions $\bar{\sigma}_x$ and $\bar{\sigma}_{xy}$ satisfying the constraint (9) have zero moment and shear force, their effect will be confined to a narrow zone of width c

adjacent to the edge of the shell. Our problem concerns a second boundary layer within which the effect of the applied bending moment and shear force is significant. We assume the width of this second layer is b and use this characteristic length b and a reference stress σ_0 related to the magnitude of M_0 or Q_0 to introduce the following nondimensional quantities:

$$\xi = \frac{x-l}{b}, \qquad \eta = \frac{y}{c}, \qquad \rho = \frac{c}{b}, \qquad \lambda = \frac{b}{l}, \tag{16}$$

$$\left[s_\xi, s_\theta, \rho^2 s_\eta, \rho s_{\xi\eta}\right] = \frac{\left[\sigma_x, \sigma_\theta, \sigma_y, \sigma_{xy}\right]}{\sigma_0}, \tag{17}$$

$$\left[u, \rho^{-1} w\right] = \left[u_x, u_y\right] \frac{E}{(1-\nu^2)\sigma_0 b}. \tag{18}$$

Here the nondimensional coordinates (ξ, η), the stress components $(s_\xi, s_\theta, s_\eta, s_{\xi\eta})$, and the displacement components (u, w) are of the order of unity. Furthermore, the parameters ρ and λ satisfy inequalities

$$0 < \rho \ll 1, \qquad 0 < \lambda \le 1. \tag{19}$$

Introducing the nondimensional quantities into the equations (4)–(6), we notice that

$$b = pl \tan\alpha = \sqrt{cR_0} \tag{20}$$

in order to obtain the classical conical shell theory as a first approximation, neglecting higher order terms in ρ. The parameter λ can be expressed as

$$\lambda = b/l = \rho \tan\alpha = \sqrt{c/R_0}\,\tan\alpha. \tag{21}$$

For small values of ρ, the characteristic length b approaches l only when α is close to 90°.

The equations of elasticity come out to be

$$\begin{aligned} &\left[(1+\lambda\xi+\rho^2\eta)s_\xi\right]_{,\xi} + \left[(1+\lambda\xi+\rho^2\eta)s_{\xi\eta}\right]_{,\eta} - \lambda s_\theta = 0, \\ &\left[(1+\lambda\xi+\rho^2\eta)s_{\xi\eta}\right]_{,\xi} + \left[(1+\lambda\xi+\rho^2\eta)s_\eta\right]_{,\eta} - s_\theta = 0 \end{aligned} \tag{22}$$

and

$$\begin{aligned} &u_{,\xi} = \frac{s_\xi - \nu s_\theta - \nu\rho^2 s_\eta}{1-\nu^2}, \qquad \frac{w+\lambda u}{1+\lambda\xi+\rho^2\eta} = \frac{s_\theta - \nu s_\xi - \nu\rho^2 s_\eta}{1-\nu^2}, \\ &w_{,\eta} = \rho^2 \frac{\rho^2 s_\eta - \nu(s_\xi + s_\theta)}{1-\nu^2}, \qquad w_{,\xi} + u_{,\eta} = 2\rho^2 \frac{s_{\xi\eta}}{1+\nu}. \end{aligned} \tag{23}$$

The stress resultants and moments are given by

$$\begin{aligned} N_x &= \sigma_0 c n_\xi, & N_\theta &= \sigma_0 c n_\theta, \\ M_x &= \sigma_0 c^2 m_\xi, & M_\theta &= \sigma_0 c^2 m_\theta, \\ & & Q_x = \sigma_0 \rho c q_\xi, \end{aligned} \tag{24}$$

where

$$\begin{aligned} n_\xi &= \int_{-1}^{1} s_\xi \frac{1+\lambda\xi+\rho^2\eta}{1+\lambda\xi}\, d\eta, & n_\theta &= \int_{-1}^{1} s_\theta\, d\eta, \\ m_\xi &= \int_{-1}^{1} s_\xi \frac{1+\lambda\xi+\rho^2\eta}{1+\lambda\xi}\eta\, d\eta, & m_\theta &= \int_{-1}^{1} s_\theta \eta\, d\eta, \\ q\xi &= \int_{-1}^{1} s_{\xi\eta} \frac{1+\lambda\xi+\rho^2\eta}{1+\lambda\xi}\, d\eta. \end{aligned} \tag{25}$$

The two sets of boundary conditions in Equations (14) and (15) can be written as

$$\xi = 0: \qquad \left[m_\xi, q_\xi, n_\xi\right] = [1,0,0], \qquad \sigma_0 = M_0/c^2, \tag{26a}$$

$$\xi = 0: \qquad \left[m_\xi, q_\xi, n_\xi\right] = [0,1,\lambda], \qquad \sigma_0 = Q_0/\rho c. \tag{26b}$$

As has been done for cylindrical shells in [1], we may expand all functions $f(\xi,\eta;\rho)$ in asymptotic series

$$f(\xi,\eta;\rho) = f_0(\xi,\eta) + \rho^2 f_1(\xi,\eta) + \cdots . \tag{27}$$

The systems of equations corresponding to the first two terms in such an expansion, along with the appropriate boundary conditions, are listed below:

$$\begin{aligned} &\left[(1+\lambda\xi)s_{\xi 0}\right]_{,\xi} + (1+\lambda\xi)s_{\xi\eta 0,\eta} - \lambda s_{\theta 0} = 0, \\ &\left[(1+\lambda\xi)s_{\xi\eta 0}\right]_{,\xi} + (1+\lambda\xi)s_{\eta 0,\eta} - s_{\theta 0} = 0, \\ &u_{0,\xi} = \frac{s_{\xi 0} - \nu s_{\theta 0}}{1-\nu^2}, \qquad \frac{w_0 + \lambda u_0}{1+\lambda\xi} = \frac{s_{\theta 0} - \nu s_{\xi 0}}{1-\nu^2}, \\ &w_{0,\eta} = 0, \qquad w_{0,\xi} + u_{0,\eta} = 0, \end{aligned} \tag{28}$$

$$s_{\xi\eta 0}(\xi,\pm 1) = 0, \qquad s_{\eta 0}(\xi,\pm 1) = 0; \tag{29}$$

$$\xi = 0: \qquad \int_{-1}^{1}\left[\eta s_{\xi 0}, s_{\xi\eta 0}, s_{\xi 0}\right] d\eta = [1,0,0] \tag{30}$$

$$\xi = 0: \qquad \int_{-1}^{1} \left[\eta s_{\xi 0}, s_{\xi\eta 0}, s_{\xi 0}\right] d\eta = [0,1,\lambda]; \tag{31}$$

$$\left[(1+\lambda\xi)s_{\xi 1}\right]_{,\xi} + (1+\lambda\xi)s_{\xi\eta 1,\eta} - \lambda s_{\theta 1} = -\eta s_{\xi 0,\xi} - \left(\eta s_{\xi\eta 0}\right)_{,\eta},$$

$$\left[(1+\lambda\xi)s_{\xi\eta 1}\right]_{,\xi} + (1+\lambda\xi)s_{\eta 1,\eta} - s_{\theta 1} = -\eta s_{\xi\eta 0,\xi} - \left(\eta s_{\xi\eta 0}\right)_{,\eta},$$

$$u_{1,\xi} = \frac{s_{\xi 1} - \nu s_{\theta 1} - \nu s_{\eta 0}}{1-\nu^2}, \qquad \frac{w_1 + \lambda u_1}{1+\lambda\xi} = \frac{s_{\theta 1} - \nu s_{\xi 1} - \nu s_{\eta 0}}{1-\nu^2} - \eta\frac{w_0 + \lambda u_0}{(1+\lambda\xi)^2},$$

$$w_{1,\eta} = -\frac{\nu}{1-\nu^2}\left(s_{\xi 0} + s_{\theta 0}\right), \qquad w_{1,\xi} + u_{1,\eta} = \frac{2}{1+\nu}s_{\xi\eta 0}, \tag{32}$$

$$s_{\xi\eta 1}(\xi, \pm 1) = 0, \qquad s_{\eta 1}(\xi, \pm 1) = 0; \tag{33}$$

$$\xi = 0: \qquad \int_{-1}^{1} \left[\eta s_{\xi 1}, s_{\xi\eta 1}, s_{\xi 1}\right] d\eta = \int_{-1}^{1} \left[\eta^2 s_{\xi 0}, \eta s_{\xi\eta 0}, 0\right] d\eta \tag{34}$$

or

$$\xi = 0: \qquad \int_{-1}^{1} \left[\eta s_{\xi 1}, s_{\xi\eta 1}, s_{\xi 1}\right] d\eta = [0,1,\lambda]. \tag{35}$$

The above systems of equations can be solved sequentially to obtain the elementary theory solution and corrections of order ρ^2. Here, we assume ρ^2 is sufficiently small and confine ourselves to the zeroth order system of equations.

Steep conical shells

We will explore the possibility of expanding the system of equations (28)–(31) in terms of the parameter $\lambda = \rho \tan\alpha$. In fact each system of equations obtained using the asymptotic expansion in ρ can be further expanded in terms of the parameter λ. The advantage of such expansions is that the governing differential equations come out to have constant coefficients in contrast with the variable coefficient type encountered in the elementary conical shell theory. Thus, in place of Kelvin functions we will have exponentials as solutions. We note that except for the case of $\tan\alpha \sim (1/\rho)$, which represents an extremely shallow shell, linear terms in λ are sufficient to retain the accuracy of the elementary theory. For convenience, we omit the subscript 0 in the zeroth order system and expand all the variables in the form

$$f(\xi, \eta; \lambda) = f_0(\xi, \eta) + \lambda f_1(\xi, \eta) + \cdots. \tag{36}$$

Equations (28)–(31) become

$$s_{\xi 0,\xi} + s_{\xi\eta 0,\eta} = 0,$$

$$s_{\xi\eta 0,\xi} + s_{\eta 0,\eta} - s_{\theta 0} = 0,$$

$$s_{\xi 0} = u_{0,\xi} + \nu w_0, \qquad s_{\theta 0} = w_0 + \nu u_{0,\xi},$$

$$w_{0,\eta} = 0, \qquad w_{0,\xi} + u_{0,\eta} = 0, \tag{37}$$

$$s_{\eta 0}(\xi, \pm 1) = 0, \qquad s_{\xi\eta 0}(\xi, \pm 1) = 0, \tag{38}$$

$$\xi = 0: \qquad \int_{-1}^{1} \left[\eta s_{\xi 0}, s_{\xi\eta 0}, s_{\xi 0}\right] d\eta = [1,0,0] \tag{39}$$

or

$$\xi = 0: \qquad \int_{-1}^{1} \left[\eta s_{\xi 0}, s_{\xi\eta 0}, s_{\xi 0}\right] d\eta = [0,1,0]. \tag{40}$$

We observe that these equations are the same as those describing a thin cylindrical shell [1] and the effects of the conicity appear in the higher order terms. The first order system is given by

$$s_{\xi 1,\xi} + s_{\xi\eta 1,\eta} = s_{\theta 0} - s_{\xi 0},$$

$$s_{\xi\eta 1,\xi} + s_{\eta 1,\eta} - s_{\theta 1} = s_{\xi\eta 0} - \xi s_{\theta 0},$$

$$s_{\xi 1} = u_{1,\xi} + \nu(w_1 + u_0 - \xi w_0), \qquad s_{\theta 1} = w_1 + u_0 + \xi w_0 - \nu u_{1,\xi},$$

$$w_1, \eta = 0, \qquad w_{1,\xi} + u_{1,\eta} = 0, \tag{41}$$

$$s_{\eta 1}(\xi, \pm 1) = 0, \qquad s_{\xi\eta 1}(\xi, \pm 1) = 0, \tag{42}$$

$$\xi = 0: \qquad \int_{-1}^{1} \left[\eta s_{\xi 1}, s_{\xi\eta 1}, s_{\xi 1}\right] d\eta = [0,0,0], \tag{43}$$

or

$$\xi = 0: \qquad \int_{-1}^{1} \left[\eta s_{\xi 1}, s_{\xi\eta 1}, s_{\xi 1}\right] d\eta = [0,0,1]. \tag{44}$$

If $\tan\alpha$ is of the order of unity, the error involved in neglecting the λ^2-terms is of the same order as the error due to neglecting the ρ^2-terms.

The solutions of the zeroth order system is identical to those of the cylindrical shell. In terms of the midplane tangential displacement $U_0(\xi)$ and normal displacement $W_0(\xi)$, the solutions of the zeroth order system are

$$
\begin{aligned}
&w_0 = W_0, \qquad u_0 = U_0 - \eta W_0', \\
&s_{\xi 0} = -\eta W_0'', \qquad s_{\theta 0} = (1-\nu^2)W_0 - \nu\eta W_0'', \\
&s_{\eta 0} = \frac{\eta^2-1}{2}\left[(1-\nu^2)\eta W_0 - \nu W_0''\right], \\
&s_{\xi\eta 0} = \frac{\eta^2-1}{2} W_0''',
\end{aligned}
\tag{45a}
$$

where ( )′ represents differentiation with respect to ξ,

$$
U_0 = \frac{1}{3}\frac{\nu}{1-\nu^2} W_0''', \tag{45b}
$$

and W_0 satisfies the differential equation

$$
W_0'''' + 4\kappa^4 W_0 = 0, \qquad 4\kappa^4 \equiv 3(1-\nu^2). \tag{45c}
$$

It remains to solve Equation (45c) subject to either one of the two sets of boundary conditions. Before we do that, let us obtain the general solution for the first order system.

Again, in terms of function U_1 and W_1 we obtain

$$
\begin{aligned}
&w_1 = W_1, \qquad u_1 = U_1 - \eta W_1', \\
&s_{\xi 1} = -\eta(W_1'' + \nu W_0') - \frac{1}{3}W_0''', \\
&s_{\theta 1} = (1-\nu^2)(W_1 - \xi W_0) - \eta(W_0' + \nu W_1''), \\
&s_{\eta 1} = \frac{\eta^2-1}{2}(W_1''' + W_0''), \\
&s_{\xi\eta 1} = \frac{\eta^2-1}{2}\left[(1-\nu^2)\eta(W_1 - 2\xi W_0) - \nu W_1'' - W_0' + \nu\xi W_0''\right],
\end{aligned}
\tag{46}
$$

where

$$
U_1 = \frac{1}{3}\frac{1}{1-\nu^2}\left[\nu W_1''' - \nu(\xi W_0'')' - (1-\nu)W_0''\right] \tag{47}
$$

and W_1 satisfies the differential equation

$$
W_1'''' + 4\kappa^4 W_1 = -2(\xi W_0''')'. \tag{48}
$$

The solutions of Equations (45c) and (48) which are bounded as $\xi \to -\infty$ are given by

$$W_0 = [A_0 \cos \kappa\xi + B_0 \sin \kappa\xi] e^{\kappa\xi}, \tag{49}$$

$$W_1 = [A_1 \cos \kappa\xi + B_1 \sin \kappa\xi] e^{\kappa\xi} + \tfrac{1}{4}(\xi W_0 - \xi^2 W_0'), \tag{50}$$

where the constants A_0, B_0, A_1, and B_1 are to be evaluated using Equations (39) and (43) when the edge moment is prescribed and using Equations (40) and (44) when the edge shear force is prescribed. In the case of prescribed moments we find

$$\begin{aligned} A_0 &= -\frac{3}{4\kappa^2}, & B_0 &= -\frac{3}{4\kappa^2}, \\ A_1 &= \frac{3}{4\kappa^3}\frac{4\nu - 1}{4}, & B_1 &= \frac{3}{4\kappa^3}\nu. \end{aligned} \tag{51}$$

In the case of prescribed shear force we have

$$\begin{aligned} A_0 &= \frac{3}{4\kappa^3}, & B_0 &= 0, \\ A_1 &= -\frac{3}{4\kappa^4}\frac{\nu}{2}, & B_1 &= -\frac{3}{4\kappa^4}\frac{\nu}{2}. \end{aligned} \tag{52}$$

With the above constants we may evaluate the stresses and displacements in the shell, including the effects of conicity.

Flexibility coefficients for steep shells

We are interested in obtaining effective normal displacement and rotation at the edge of the shell and in relating these quantities to the applied moment and shear force. In order to account for the zero axial force, we introduce weighted axial displacement U_z^*, radial displacement U_r^*, and rotation β^* through the work relation

$$\begin{aligned} &U_z^*(N_x \cos\alpha - Q_x \sin\alpha) + U_r^*(N_x \sin\alpha + Q_x \cos\alpha) + \beta^* M_x \\ &\quad = \frac{1}{2}\int_{-c}^{c} (u_x \sigma_x + u_y \sigma_{xy})\left(1 + \frac{y}{R_0}\right) dy, \end{aligned} \tag{53}$$

where all the quantities are evaluated at $x = l$. Using the relation $N_x = Q_x \tan\alpha$, Equation (53) reduces to

$$W^* Q_0 + \beta^* M_0 = \frac{1}{2}\int_{-c}^{c} (u_x \sigma_x + u_y \sigma_{xy})\left(1 + \frac{y}{R_0}\right) dy, \tag{54}$$

where the effective edge displacement W^* is defined as $U_r^*/\cos\alpha$.

We introduce the flexibility coefficients C_{MM}, $C_{MQ}=C_{QM}$, and C_{QQ} through the relations

$$\begin{aligned} \beta^* &= C_{MM}M_0 + C_{MQ}Q_0, \\ W^* &= C_{QM}M_0 + C_{QQ}Q_0. \end{aligned} \tag{55}$$

Differentiating Equation (55) with respect to M_0 and Q_0 and identifying the coefficients of M_0 and Q_0 on the right hand side, we get

$$\begin{aligned} C_{MM} &= \int_{-c}^{c} [u_{x,m}\sigma_{x,m} + u_{y,m}\sigma_{xy,m}]\left(1+\frac{y}{R_0}\right) dy, \\ C_{MQ} &= \int_{-c}^{c} [u_{x,m}\sigma_{x,q} + u_{y,m}\sigma_{xy,q}]\left(1+\frac{y}{R_0}\right) dy, \\ C_{QQ} &= \int_{-c}^{c} [u_{x,q}\sigma_{x,q} + u_{y,q}\sigma_{xy,q}]\left(1+\frac{y}{R_0}\right) dy, \end{aligned} \tag{56}$$

where we have used the notation

$$f_{,m} = f(M_0=1,\, Q_0=0), \qquad f_{,q} = f(M_0=0,\, Q_0=1). \tag{57}$$

In terms of the nondimensional variables the flexibility coefficients are

$$\begin{aligned} C_{MM} &= \frac{4}{3}\frac{\kappa^4}{Ec^2\rho}\int_{-1}^{1} [u_{\xi,m}s_{\xi,m} + u_{\eta,m}s_{\xi\eta,m}]\left(1+\frac{\rho^2\eta}{1+\lambda\xi}\right) d\eta, \\ C_{MQ} &= \frac{4}{3}\frac{\kappa^4}{Ec\rho^2}\int_{-1}^{1} [u_{\xi,q}s_{\xi,m} + u_{\eta,q}s_{\xi\eta,m}]\left(1+\frac{\rho^2\eta}{1+\lambda\xi}\right) d\eta, \\ C_{QQ} &= \frac{4}{3}\frac{\kappa^4}{E\rho^3}\int_{-1}^{1} [u_{\xi,q}s_{\xi,q} + u_{\eta,q}s_{\xi\eta,q}]\left(1+\frac{\rho^2\eta}{1+\lambda\xi}\right) d\eta. \end{aligned} \tag{58}$$

Neglecting the ρ^2 and λ^2 terms and introducing

$$\begin{aligned} u_{\xi 0} &= U_0 - \eta W_0', & u_{\xi 1} &= U_1 - \eta W_1' \\ u_{\eta 0} &= W_0, & u_{\eta 1} &= W_1 \end{aligned} \tag{59}$$

and

$$\begin{aligned} &s_{\xi 0} = \tfrac{3}{2}\eta, \quad s_{\xi 1} = s_{\xi\eta 0} = s_{\xi\eta 1} = 0 \qquad \text{when} \quad M_0 = 1, \quad Q_0 = 0, \\ &s_{\xi 0} = 0, \quad s_{\xi 1} = \tfrac{1}{2}, \quad s_{\xi\eta 0} = \tfrac{3}{4}(1-\eta^2), \quad s_{\xi\eta 1} = 0 \qquad \text{when} \quad M_0 = 0, \quad Q_0 = 1, \end{aligned} \tag{60}$$

we evaluate the above integrals to obtain

$$C_{MM} = \frac{2\kappa^3}{Ec^2\rho}\left(1-\frac{\lambda}{\kappa}\frac{4\nu-1}{2}\right),$$

$$C_{MQ} = -\frac{\kappa^2}{Ec\rho^2}\left(1-\frac{\lambda}{\kappa}\frac{4\nu-1}{4}\right), \tag{61}$$

$$C_{QQ} = \frac{\kappa}{E\rho^3}\left(1-\frac{\lambda}{\kappa}\nu\right).$$

We note that β^* is proportional to $-W'$, and W^* is proportional to $W+\lambda U$, with the constants of proportionality representing the factors used in the nondimensionalization.

Shallow conical shells

It is of interest to seek asymptotic expansions of the zeroth order system (28)–(31) in terms a small parameter when the semiapex angle α is close to 90°. For this purpose we introduce the following nondimensional quantities:

$$\xi = \frac{x}{l}, \qquad \eta = \frac{y}{c}, \qquad \delta = \frac{c}{l}, \qquad \Lambda = \frac{\cot\alpha}{\delta}, \tag{62}$$

$$\left[s_\xi, s_\theta, \delta^2 s_\eta, \delta s_{\xi\eta}\right] = \frac{\left[\sigma_x, \sigma_\theta, \sigma_y, \sigma_{xy}\right]}{\sigma_0}, \tag{63}$$

$$\left[u, \delta^{-1}w\right] = \frac{E}{(1-\nu^2)\sigma_0 l}\left[u_x, u_y\right]. \tag{64}$$

Here the nondimensional coordinates (ξ,η), stress components $(s_\xi, s_\theta, s_\eta, s_{\xi\eta})$, and displacement components (u, w) are of the order of unity.

The stress resultants and moments are given by

$$N_x = \sigma_0 c n_\xi, \qquad N_\theta = \sigma_0 c n_\theta,$$

$$M_x = \sigma_0 c^2 m_\xi, \qquad M_\theta = \sigma_0 c^2 m_\theta, \tag{65}$$

$$Q_x = \sigma_0 \delta c q_\xi,$$

where

$$n_\xi = \int_{-1}^{1} s_\xi\left(1+\frac{\Lambda\delta^2\eta}{\xi}\right)d\eta, \qquad n_\theta = \int_{-1}^{1} s_\theta\, d\eta,$$

$$m_\xi = \int_{-1}^{1} s_\xi\left(1+\frac{\Lambda\delta^2\eta}{\xi}\right)\eta\, d\eta, \qquad m_\theta = \int_{-1}^{1} s_\theta\, d\eta, \tag{66}$$

$$q_\xi = \int_{-1}^{1} s_{\xi\eta}\left(1+\frac{\Lambda\delta^2\eta}{\xi}\right)d\eta.$$

The two sets of boundary conditions in Equations (14) and (15) can be written as

$$\xi = 0: \quad \left[m_\xi, q_\xi, n_\xi\right] = [1,0,0], \qquad \sigma_0 = M_0/c^2; \tag{67}$$

$$\xi = 0: \quad \left[m_\xi, q_\xi, n_\xi\right] = [0,\Lambda,1], \qquad \sigma_0 = Q_0/\delta c. \tag{68}$$

The classical equations of conical shells are now obtained as

$$\begin{gathered}
\left(\xi s_\xi\right)_{,\xi} + \xi s_{\xi\eta,\eta} - s_\theta = 0, \\
\left(\xi s_{\xi\eta}\right)_{,\xi} + \xi s_{\eta,\eta} - \Lambda s_\theta = 0, \\
u_{,\xi} = \frac{s_\xi - \nu s_\theta}{1-\nu^2}, \qquad \frac{u+\Lambda w}{\xi} = \frac{s_\theta - \nu s_\xi}{1-\nu^2}, \\
w_{,\eta} = 0, \qquad w_{,\xi} + u_{,\eta} = 0,
\end{gathered} \tag{69}$$

$$s_{\xi\eta}(\xi, \pm 1) = 0, \qquad s_\eta(\xi, \pm 1) = 0, \tag{70}$$

$$\xi = 0: \quad \int_{-1}^{1} \left[\eta s_\xi, s_{\xi\eta}, s_\xi\right] d\eta = [1,0,0] \tag{71}$$

or

$$\xi = 0: \quad \int_{-1}^{1} \left[\eta s_\xi, s_{\xi\eta}, s_\xi\right] d\eta = [0,\Lambda,1]. \tag{72}$$

We note that the omitted terms in the above system are of the order of $\delta \cot \alpha$. For converting the above system into a sequence of equations with constant coefficients we use the parameter $\Lambda = (\cot \alpha)/\delta$. For Λ to be small $\cot \alpha$ must be of the order of δ^2, and then the neglected terms in the equations of elasticity turn out to be of the order of δ^3. This implies that we may consider relatively thick shells. For example, if $\delta = 0.2$, the error involved in the classical equations of conical shells comes out to be of the order of 0.008, and for $\Lambda = 0.2$ we may have $\alpha > 88°$. We need only terms of order Λ^2 in an asymptotic expansion in terms of Λ to retain the same accuracy as in the classical shell equations. Of course, expanding the classical equations in terms of Λ is equivalent to perturbing a circular plate into a shallow cone:

$$\begin{gathered}
\left(\xi s_{\xi 0}\right)_{,\xi} + \xi s_{\xi\eta 0,\eta} - s_{\theta 0} = 0, \\
\left(\xi s_{\xi\eta 0}\right)_{,\xi} + \xi s_{\eta 0,\eta} = 0, \\
s_{\xi 0} = u_{0,\xi} + \frac{\nu u_0}{\xi}, \qquad s_{\theta 0} = \frac{u_0}{\xi} + \nu u_{0,\xi}, \\
w_{0,\eta} = 0, \qquad w_{0,\xi} + u_{0,\eta} = 0,
\end{gathered} \tag{73}$$

$$s_{\eta 0}(\xi, \pm 1) = 0, \qquad s_{\xi\eta 0}(\xi, \pm 1) = 0, \tag{74}$$

$$\xi = 0: \quad \int_{-1}^{1} \left[\eta s_{\xi 0}, s_{\xi\eta 0}, s_{\xi 0}\right] d\eta = [1,0,0] \tag{75}$$

or

$$\xi = 0: \quad \int_{-1}^{1} \left[\eta s_{\xi 0}, s_{\xi\eta 0}, s_{\xi 0}\right] d\eta = [0,0,1]. \tag{76}$$

We observe that these equations are the same as those describing a circular plate and the effects of the conicity appear in the higher order equations. The first and second order systems are given by

$$\left(\xi_{\xi 1}\right)_{,\xi} + \xi s_{\xi\eta 1,\eta} = s_{\theta 0},$$

$$\left(\xi s_{\xi\eta 1}\right)_{,\xi} + \xi s_{\eta 1,\eta} = 0,$$

$$s_{\xi 1} = u_{1,\xi} + \nu\frac{u_1 + w_0}{\xi}, \qquad s_{\theta 1} = \frac{u_1 + w_0}{\xi} + \nu u_{1,\xi},$$

$$w_{1,\eta} = 0, \qquad w_{1,\xi} + u_{1,\eta} = 0, \tag{77}$$

$$s_{\eta 1}(\xi, \pm 1) = 0, \qquad s_{\xi\eta 1}(\xi, \pm 1) = 0, \tag{78}$$

$$\xi = 0: \quad \int_{-1}^{1} \left[\eta s_{\xi 1}, s_{\xi\eta 1}, s_{\xi 1}\right] d\eta = [0,0,0] \tag{79}$$

or

$$\xi = 0: \quad \int_{-1}^{1} \left[\eta s_{\xi 1}, s_{\xi\eta 1}, s_{\xi 1}\right] d\eta = [0,1,0]; \tag{80}$$

$$\left(\xi s_{\xi 2}\right)_{,\xi} + \xi s_{\xi\eta 2,\eta} = s_{\theta 1},$$

$$\left(\xi s_{\xi\eta 2}\right)_{,\xi} + \xi s_{\eta 2,\eta} = 0,$$

$$s_{\xi 2} = u_{2,\xi} + \nu\frac{u_2 + w_1}{\xi}, \qquad s_{\theta 2} = \frac{u_2 + w_1}{\xi} + \nu u_{2,\xi},$$

$$w_2, \eta = 0, \qquad w_{2,\xi} + u_{2,\eta} = 0, \tag{81}$$

$$s_{\eta 2}(\xi, \pm 1) = 0, \qquad s_{\xi\eta 2}(\xi, \pm 1) = , \tag{82}$$

$$\xi = 0: \quad \int_{-1}^{1} \left[\eta s_{\xi 2}, s_{\xi\eta 2}, s_{\xi 2}\right] d\eta = [0,0,0] \tag{83}$$

or

$$\xi = 0: \quad \int_{-1}^{1} \left[\eta s_{\xi 2}, s_{\xi\eta 2}, s_{\xi 2}\right] d\eta = [0,0,0]. \tag{84}$$

The displacement components u_i and w_i ($i = 0,1,2$) may be expressed in terms of midplane tangential and normal displacements $U_i(\xi)$ and $W_i(\xi)$ in the form

$$w_i = W_i, \qquad u_i = U_i - \eta W_i' \tag{85}$$

to have vanishing normal and shear strains. Introducing these into the constitutive relations, we find

$$
\begin{aligned}
s_{\xi i} &= U_i' + \nu\frac{U_i+W_i}{\xi} - \eta\left(W_i'' + \nu\frac{W_i'}{\xi}\right),\\
s_{\theta i} &= \frac{U_i+W_i}{\xi} + \nu\frac{U_i'}{\xi} - \eta\left(\nu W_i'' + \frac{W_i'}{\xi}\right),\\
\xi s_{\xi\eta i} &= \frac{\eta^2-1}{2}\left((\xi W_i'')' + \frac{W_i'}{\xi}\right),\\
\xi s_{\eta i} &= \frac{1-\eta^2}{2}\left[\left(\frac{W_{i-1}'}{\xi} + \nu W_{i-1}''\right) + \frac{\eta}{3}\left((\xi W_i'')' + \frac{W_i'}{\xi}\right)'\right],
\end{aligned}
\tag{86}
$$

where quantities with negative subscripts are taken to be zeros.

The boundary conditions at $\eta = \pm 1$ yield the differential equations for U_i and W_i,

$$
\begin{aligned}
(\xi W_i'')'' - \left(\frac{W_i'}{\xi}\right)' &= 3\left(\frac{U_{i-1}+W_{i-2}}{\xi} + \nu U_{i-1}'\right),\\
(\xi U_i')' - \frac{U_i}{\xi} &= \frac{W_{i-1}}{\xi} - \nu W_{i-1}'.
\end{aligned}
\tag{87}
$$

For $i = 0$, 1, and 2 the solutions of the differential equations (86) and (87) can be written as

$$
\begin{aligned}
W_i &= \frac{1}{2}\frac{C_i\xi^2}{1+\nu} - \frac{1-\nu}{32}C_{i-2}\xi^4 - \tfrac{1}{3}A_{i-1}\xi^3,\\
U_i &= \frac{A_i\xi}{1+\nu} - \frac{1-3\nu}{24}A_{i-2}\xi^3 + \frac{1}{6}\frac{1-2\nu}{1+\nu}C_{i-1}\xi^2,
\end{aligned}
\tag{88}
$$

where C_i and A_i are constants to be evaluated. We also note that the above solutions do not include certain terms which have singularities as $\xi \to 0$.

The stresses can be obtained in terms of these constants as

$$
\begin{aligned}
s_{\xi i} &= A_i - \frac{1-\nu^2}{8}A_{i-2}\xi^2 + \frac{1-\nu}{3}C_{i-1}\xi\\
&\quad - \eta\left[C_i - (1-\nu)\frac{3+\nu}{8}C_{i-2}\xi^2 - (2+\nu)A_{i-1}\xi\right],\\
s_{\theta i} &= A_i - 3\frac{1-\nu^2}{8}A_{i-2}\xi^2 + 2\frac{1-\nu}{3}C_{i-1}\xi\\
&\quad - \eta\left[C_i - (1-\nu)\frac{1+3\nu}{8}C_{i-2}\xi^2 - (1+2\nu)A_{i-1}\xi\right],\\
\xi s_{\xi\eta i} &= \frac{1-\eta^2}{2}\left[3A_{i-1}\xi + (1-\nu)C_{i-2}\xi^2\right],\\
\xi s_{\eta i} &= \frac{1-\eta^2}{2}\left[C_{i-1} - (1+2\nu)A_{i-2}\xi\right] + \eta\frac{\eta^2-1}{6}\left[3A_{i-1} + 2(1-\nu)C_{i-2}\xi\right].
\end{aligned}
\tag{89}
$$

The stress resultants and moments are given by

$$
\begin{aligned}
m_{\xi i} &= \frac{2}{3}\left[C_i - (1-\nu)\frac{3+\nu}{8}C_{i-2}\xi^2 - (2+\nu)A_{i-1}\xi\right],\\
n_{\xi i} &= 2\left[A_i - \frac{1-\nu^2}{8}A_{i-2}\xi^2 + \frac{1-\nu}{3}C_{i-1}\xi\right],\\
q_{\xi i} &= \tfrac{2}{3}\left[3A_{i-1} + (1-\nu)C_{i-2}\xi\right].
\end{aligned}
\tag{90}
$$

The overall equilibrium condition $q_{\xi i} = n_{\xi i-1}$ is, of course, satisfied by the above resultants.

We next evaluate the constants in these solutions using the two systems of boundary conditions. In the case of prescribed moment we find

$$
\begin{aligned}
A_0 &= 0, \qquad A_1 = \frac{1-\nu}{2}, \qquad A_2 = 0,\\
C_0 &= -\tfrac{3}{2}, \qquad C_1 = 0, \qquad C_2 = \frac{7-2\nu-5\nu^2}{16},
\end{aligned}
\tag{91}
$$

and in the case of prescribed shear force

$$
A_0 = \tfrac{1}{2}, \qquad A_1 = 0, \qquad A_2 = -\frac{5-8\nu-5\nu^2}{48},
\tag{92}
$$

$$
C_0 = 0, \qquad C_1 = \frac{2+\nu}{2}, \qquad C_2 = 0.
\tag{93}
$$

With the above constants we may evaluate the stresses and displacements in the shell for small values of Λ.

Flexibility coefficients for shallow shells

As for steep shells the flexibility coefficients can be expressed as

$$
\begin{aligned}
C_{MM} &= \frac{1-\nu^2}{Ec^2\delta}\int_{-1}^{1}\left(u_{\xi,m}s_{\xi,m} + u_{\eta,m}s_{\xi\eta,m}\right)\left(1+\delta^2\Lambda\eta\right)d\eta,\\
C_{MQ} &= \frac{1-\nu^2}{Ec\delta}\int_{-1}^{1}\left(u_{\xi,q}s_{\xi,m} + u_{\eta,q}s_{\xi\eta,m}\right)\left(1+\delta^2\Lambda\eta\right)d\eta,\\
C_{QQ} &= \frac{1-\nu^2}{E\Lambda^2\delta}\int_{-1}^{1}\left(u_{\xi,q}s_{\xi,q} + u_{\eta,q}s_{\xi\eta,q}\right)\left(1+\delta^2\Lambda\eta\right)d\eta.
\end{aligned}
\tag{94}
$$

Neglecting the $\delta^2\Lambda$ and Λ^3 terms, we find

$$C_{MM} = \frac{1-\nu^2}{Ec^2\delta}\frac{3}{2(1+\nu)}\left(1-\Lambda^2\frac{1-\nu}{4}\right),$$

$$C_{MQ} = -\frac{1-\nu^2}{Ec\delta}\frac{1}{2(1+\nu)}\left[1+O(\Lambda^2)\right], \tag{95}$$

$$C_{QQ} = \frac{1-\nu^2}{E\Lambda^2\delta}\frac{1}{2(1+\nu)}\left(1+\Lambda^2\frac{5+\nu}{12}\right).$$

We note that it would have been more appropriate to obtain influence coefficients involving N_0 instead of Q_0 for this problem. However, a comparison with the results of thin conical shell theory is facilitated by employing C_{MQ} and C_{QQ}.

Exact solution and comparison

Exact solutions of the thin shell equations (28)–(31) are well known. It is of interest to compare our present results with the exact results. A brief outline of the solution in terms of Kelvin functions is given below:

$$w = W, \qquad u = U - \eta\mathscr{D}W,$$

$$s_\xi = \mathscr{D}U - \eta\mathscr{D}^2W + \nu\frac{U+\Lambda W-\eta\mathscr{D}W}{\xi},$$

$$s_\theta = \nu\left(\mathscr{D}U-\eta\mathscr{D}^2W\right) + \frac{U+\Lambda W-\eta\mathscr{D}W}{\xi}, \tag{96}$$

$$s_{\xi\eta} = \frac{\eta^2-1}{2\xi^2}\left[(\xi\mathscr{D})^2-1\right]\mathscr{D}W,$$

$$s_\eta = \frac{\eta-\eta^3}{6\xi^2}\mathscr{D}\frac{1}{\xi}\left[(\xi\mathscr{D})^2-1\right]\mathscr{D}W - \Lambda\frac{\eta^2-1}{2\xi^2}\left[1+\nu\xi\mathscr{D}\right]\mathscr{D}W,$$

where $\mathscr{D}$ is the differential operator $\partial/\partial\xi$ and the functions U and W satisfy

$$\left[(\xi\mathscr{D})^2-1\right]U + \Lambda(\nu\xi\mathscr{D}-1)W = 0,$$

$$\Lambda(\xi\mathscr{D}+\nu)U + \frac{1}{3\xi}\left[(\xi\mathscr{D})^2-1\right]\mathscr{D}W + \nu\Lambda^2W = 0. \tag{97}$$

Eliminating U from the above relations, we obtain the classical conical shell equation

$$\left(\left[(\xi\mathscr{D})^2-1\right]\frac{1}{\xi}\left[(\xi\mathscr{D})^2-1\right]\frac{1}{\xi}+\mu^4\right)\xi\mathscr{D}W = 0, \tag{98}$$

where $\mu^4 = 3(1-\nu^2)\Lambda^2$. The differential operator can be factored to obtain the solution

$$\mathscr{D}W = Av + \bar{A}\bar{v} \tag{99}$$

where v satisfies

$$\left[(\xi\mathscr{D})^2 - 1\right]v - i\mu^2\xi v = 0, \tag{100}$$

and where $\bar{\ }$ represents complex conjugate. With the change of variable

$$\zeta = 2\mu\sqrt{\xi} \tag{101}$$

the above equation can be transformed into one in the Bessel equation family,

$$\zeta^2 v'' + \zeta v' - \left(i\zeta^2 + 4\right)v = 0, \tag{102}$$

where prime denotes differentiation with respect to ζ. The solution of this equation in terms of Kelvin functions is given by

$$v = \mathrm{ber}_2(\zeta) + i\,\mathrm{bei}_2(\zeta). \tag{103}$$

The quantities of interest, M_ξ, q_ξ, $W'(\xi)$, and $W + U/\Lambda$ at the edge $\xi = 1$, can be obtained as

$$\begin{aligned} m_\xi &= -\frac{2}{3}\mathrm{Re}\left[A(v' + \nu v)\right], \\ q_\xi &= \frac{2\mu^2}{3}\mathrm{Im}\,Av, \\ W' &= \mathrm{Re}\,Av, \\ W + \frac{U}{\Lambda} &= \frac{\mathrm{Im}\left[A(v' - \nu v)\right]}{\mu^2}. \end{aligned} \tag{104}$$

For the two cases of boundary conditions we find

$$A = -\frac{3}{4}\frac{\bar{v}}{\mathrm{Re}(\bar{v}v' + \nu\bar{v}v)} \tag{105}$$

and

$$A = -\frac{3}{4i\mu^2}\frac{\bar{v}' + \nu\bar{v}}{\mathrm{Re}(\bar{v}v' + \nu\bar{v}v)}. \tag{106}$$

Using the notation

$$p = \bar{v}v, \qquad q = \mathrm{Im}\,v'\bar{v}, \qquad r = \mathrm{Re}\,v'\bar{v}, \qquad s = v'\bar{v}', \tag{107}$$

we write the flexibility coefficients as

$$\begin{aligned} C_{MM} &= \frac{3}{2}\frac{1-\nu^2}{Ec^2\delta}\frac{p}{\mu r+\nu p}, \\ C_{MQ} &= -\frac{3}{2\mu}\frac{1-\nu^2}{Ec\delta}\frac{q}{\mu r+\nu p}, \\ C_{QQ} &= \frac{3}{2\mu^4}\frac{1-\nu^2}{E\delta}\frac{\mu^2-\nu^2 p}{\mu r+\nu p}. \end{aligned} \tag{108}$$

To compare the above exact results with our asymptotic results we have taken $\nu = 0.3$ and $\delta(=c/l) = 0.05$, 0.1, and 0.2. The semiapex angle is varied from 10°

Table 1
Comparison[a] of Exact and Asymptotic Values of c_{mm} for Steep Shells

	c_{mm}				
α	$\delta = 0.01$	0.02	0.03	0.04	0.05
10°	0.069	0.098	0.119	0.138	0.154
	0.069	0.098	0.120	0.138	0.154
20°	0.099	0.140	0.171	0.197	0.220
	0.099	0.140	0.171	0.198	0.221
30°	0.125	0.176	0.215	0.247	0.276
	0.125	0.176	0.216	0.249	0.278
40°	0.150	0.211	0.258	0.297	0.330
	0.150	0.212	0.260	0.299	0.334
50°	0.178	0.251	0.306	0.351	0.391
	0.179	0.252	0.309	0.356	0.397
60°	0.215	0.301	0.366	0.420	0.467
	0.216	0.304	0.371	0.428	0.478
70°	0.269	0.376	0.456	0.673	0.717
	0.271	0.382	0.466	0.537	0.599
80°	0.381	0.679	0.761	0.824	0.875
	0.387	0.545	0.665	0.765	0.853

[a]The exact values are shown above the asymptotic ones.

Table 2
Comparison[a] of Exact and Asymptotic Values of c_{mm} for Shallow Shells

	c_{mm}		
α	$\delta = 0.05$	$\delta = 0.1$	$\delta = 0.2$
89.5°	1.152	1.153	1.154
	1.148	1.152	1.153
89.0°	1.146	1.152	1.153
	1.129	1.148	1.152
88.5°	1.136	1.149	1.153
	1.098	1.140	1.150
88.0°	1.124	1.146	1.152
	1.055	1.129	1.148
87.5°	1.109	1.141	1.150
	1.000	1.115	1.144
87.0°	1.092	1.136	1.149
	0.932	1.098	1.140
86.5°	1.074	1.130	1.148
	0.852	1.078	1.135
86.0°	1.056	1.123	1.146
	0.759	1.055	1.129
85.5°	1.038	1.116	1.144
	0.654	1.028	1.122
85.0°	1.020	1.109	1.141
	0.536	0.999	1.115

[a]The exact values are shown above the asymptotic ones.

to 80° for steep shells, and from 89.5° to 85° for shallow shells. Furthermore, we define a nondimensional coefficient

$$c_{mm} = C_{MM}\frac{Ec^2\delta}{1-\nu^2}. \tag{109}$$

The results for steep shells are shown in Table 1. As mentioned earlier, these values are surprisingly close for thin shells even when the apex angle is as high as 70°. The results for shallow shells are given in Table 2. The asymptotic solutions are good approximations only for α close to 90°.

Conclusions

Equations of thin conical shells are derived from three dimensional (rotationally symmetric) elasticity equations using a small parameter ρ, the thickness to shell characteristic length ratio. In the case of steep shells, these equations are further expanded using a parameter λ, the ratio of the characteristic length associated with edge bending to the shell length. This parameter is small even when the semiapex angle of the shell is close to 70° for thin shells. The solutions of conical shell equations can be expressed as the sum of the solutions for a cylindrical shell and small perturbations in terms of λ. In the case of shallow conical shells, the thin shell equations can be expanded in terms of a small parameter Λ [$= (\cot\alpha)l/c$]. This expansion is useful when the semiapex angle is very close to 90° and the solutions are perturbed results of circular plate theory.

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