Business Valuation Review

Volume 21, No. 2 June 2002
The Quarterly Journal of the Business Valuation Committee of the American Society of Appraisers

Problems in the QMDM and Comparison to Economic Components Model: A Response to Chris Mercer

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1. Introduction

It seems to me that healthy dialogue among practitioners is a useful tool in facilitating our growth as a profession. It is in that spirit that I wish to respond to my colleague, Chris Mercer's recent article,[2] wherein he asserts that my misunderstanding of his Quantitative Marketability Discount Model (QMDM) explains the disparity in my results and his in calculating the discount for lack of marketability (DLOM). Accordingly, in this article I will:

- Provide an Explanation of the Economic Components Model (ECM)
- Compare the Theoretical Underpinnings of ECM and QMDM
- Provide an empirical test of the QMDM vs. the ECM
- Address logical inconsistencies in Mr. Mercer's arguments
- Compare the scope of the two models and address theoretical strengths and weaknesses

This article consists of five sections, including this introduction. In Section 2, I explain the theoretical (and some empirical) basis of the ECM. In Section 3, I provide an empirical test of the two models with restricted stock data. In Section 4, I discuss inconsistencies in the QMDM, and Section 5 is my conclusion.

2. Economic Components Model (ECM)[3]

A complete presentation of ECM is too lengthy for and is outside of the scope of this article. For that, I must refer readers to Chapters 7–9 of my book, *Quantitative Business Valuation: A Mathematical Approach for Today's Professionals* (**QBV**). However, in a few pages, I can explain the logic of ECM and some of the key research that comprise its theoretical and empirical underpinnings.

It is important to understand that Section 2, almost half of the length of this article, is optional reading. While this entire section provides additional background that will enrich the reader's understanding of the debate between Mr. Mercer and myself, it is not necessary material and can be skipped. Those who wish to do so can safely skip to Section 3, the Empirical Test of the Two Models. That being said, let's move on.

The ECM contains four components that act as "building blocks" in calculating DLOM. We will discuss each of the components in its own section, although we treat the last two components together in one section.

Component #1: Delay to Sale

The base component in the ECM is one that measures the economic disadvantage of being illiquid for a material amount of time.

Psychology of Illiquidity

Before immediately jumping into the measurement of the Delay to Sale, in QBV, page 250, I cite a chapter from another book[4] that discussed the Ellsberg Paradox, developed by Dr. Daniel Ellsberg, later of Pentagon Papers fame. Ellsberg demonstrated that even if people are given the choice between two gambles of equal expected value, they still strongly prefer to take the gamble that is well defined than the one that is ill defined. The authors defined ambiguity as a second-order uncertainty, i.e., uncertainty about uncertainties.

It is my opinion that this psychological point drives much of the phenomena of control and marketability. Control affects the degree of clarity of the environment and security, on one hand, or ambiguity and lack of security, on the other, that shareholders experience, depending on the degree of their control.[5] How many minority shareholders are booted out of their firms by control shareholders after long years of service? I have testified in a number of such cases, and it is likely that many of the readers of this journal have done so, too.

In ECM, control affects marketability in each of the different components. Compared to control interests, private minority interests differ in the following ways:

(1) They should take longer to sell, as they are less desirable. That exposes the minority shareholder who wants to sell to negative (or positive) changes in value for a longer period than the control shareholder.

(2) There are fewer buyers for minority interests. As a practical matter in most privately held firms, the control shareholder is often the only feasible buyer for minority interests. Even if one can find an interested buyer for a minority interest, that poten-

tial buyer is in a much stronger negotiating position than the potential buyers for a control interest and is likely to drive down the price due to lack of competition among buyers.

(3) Transactions costs of selling minority interests in private firms usually should be higher than the transactions costs of selling control interests, as the former are less desirable interests and there are no organized markets in which to sell them.

Thus, while this is an article on DLOM, we cannot ignore the issue of control, as control affects all components of marketability. Before diving into the calculation of DLOM, we will discuss the different levels of value charts that have been proposed and described in the literature.

The Levels of Value Charts

There are essentially three different levels of value charts. The (Modified) Traditional Levels of Value Chart appears in Figure 1. It is a single column.

It is important to note that Chris Mercer's position is that control interests in private firms do not receive a discount for lack of marketability. His primary reason for that is that the control shareholder enjoys the full benefit of cash flows during the time to marketability, although he has other reasons for that conclusion.[7] I disagree with Mr. Mercer's conclusion, although a well-managed firm can do much to minimize the impact of DLOM.

Figure 1: (Modified) Traditional Levels of Value Chart

Level of Value	Adjustment Up To	Adjustment Down To
Strategic Value[6] (Synergistic Buyer)	Value of Synergies	NA
Control Value	Control Premium	Eliminate Synergies
Marketable Minority Value	Reverse out DLOM	DLOC
Private Minority Value	NA	DLOM

Michael Bolotsky's 2 x 2 Levels of Value Chart appears in Figure 2. Unlike the traditional Levels of Value Chart, where a control interest is always on top, Bolotsky argued—correctly in my opinion—that the value of control may or may not outweigh the discount for lack of marketability.

Figure 2: Michael Bolotsky's 2 x 2 Levels of Value Chart[8]

	Public	Private
Control	x	x
Minority	x	x

In QBV, page 230, I extend this levels of value chart to a 3 x 2 chart, as follows:

Figure 3: Abrams' 3 x 2 Levels of Value Chart

	Public	Private
Control	x	x
Minority (well treated)	x	x
Minority (exploited)	x	x

Well-treated minority shareholders are usually the owners of publicly traded companies, while minority shareholders in private firms may or may not be well-treated.[9] Even those who are well-treated today may not be well-treated in the future; thus it is logical that the DLOM for well-treated minority interests should bear a discount component for the fear that things could change for the worse someday—a fear about which a control shareholder would never have to worry.

The two different minority levels in my 3 x 2 chart represent extremes, and there are many gradations in between. A completely well treated minority interest would mean the control interest would be entirely benevolent, i.e., it would take only an arm's length salary for services, it would never expropriate wealth from the minority shareholders, and it would have to be so completely trustworthy that a minority shareholder would never have a passing worry that the control shareholder would change this policy in the future. That would also necessitate selling to a third party or giving as an inheritance to a child only with the same degree of benevolence, *ad infinitum*. It is questionable whether any human being can fill those shoes.

Exploited minority shareholders are well known as a phenomenon. Of course, there are degrees of exploitation, so even that one row on the chart is really a continuum. A control shareholder never has the fear that he or she will become tomorrow's exploited shareholder. However, any minority shareholder—no matter how well treated today—is likely to be concerned about the environment deteriorating in the future. This is an example of ambiguity. Because lack of control exposes one to an unknown degree of change, it contains an ambiguity that lowers the value of a minority interest.

Measuring the Delay to Sale Component

While the degree of control has a direct impact on value through the control premium or discount for lack of control, it also has a large indirect impact on value through its effect on marketability. A control interest has the power to sell the firm at any time, while minority interests can wait for decades to sell. Thus, control interests are much easier to sell and should sell faster than minority interests. The ECM and QMDM each recognize timing as a critical element in the DLOM calculations, but very differently. In the QMDM, the issue is when will the control shareholder sell the entire firm. In most privately held firms, that is usually a long-term proposition. The minority shareholder has to wait for the control shareholder to sell the entire firm to achieve liquidity. In the ECM, the ultimate timing of the "final sale" is usually not the main issue unless that sale is near at hand. The timing issue in the ECM is how long would it take for me to find a buyer for my interest even if the firm as a whole does not sell. That is the period of Illiquidity that I use in the Delay-to-Sale Component.

I measure the delay-to-sale component by estimating the restricted stock discount for a business with similar characteristics to the subject company. The logic for this is that ownership in a privately held business—whether an operating or a holding company—is similar to owning restricted stock in publicly held company.

Restricted stock is legally restricted from sale for a well-defined period of time. According to SEC Rule 144, a non-affiliate (one who is not an officer or director) can begin selling restricted stock in one year (two years before April 27, 1997) according to the SEC's "dribble out" rules and sell all stock at the end of the second year (third year before April 27, 1997). While the owner of a privately-held company may not have legal restrictions precluding immediate or near-term sale, as a practical matter it takes a long time to sell such a business interest—even if the owner has prepared the business for sale, which most have not. During the time of preparing the business for sale, engaging a business broker or investment banker, and marketing the business, there may be good news or bad news. Investors, being risk averse, are more concerned about the bad news than the good news and require a discount to compensate them for the time in which they are exposed to a change in the condition, i.e., value, of the business. I model this "Delay to Sale" component in two different ways, which I will discuss briefly here and in more detail later in the article.

The first method of modeling the restricted stock discount is David Chaffe's use of the Black-Scholes Options Pricing Model (BSOPM) to calculate the value of a put option as a proxy for the cost of illiquidity.[10] The second method of modeling the restricted stock discount is my own regression analysis of the Management Planning, Inc. data, which we will discuss in more detail later.

Component #2: Monopsony Power of the Buyers

The second component of ECM is a discount for "monopsony power". A monopsony is a single buyer. It is the mirror image of a monopoly, a single seller. As a monopoly has the power to withhold production and drive price above marginal cost, a monopsonist has the power to withhold purchase and drive price down. Most small and even medium size private businesses face few interested buyers. While they do not face literally one sole buyer, usually they do not have enough interested buyers to drive the price to its theoretical maximum in pure competition with many interested buyers.

I model this by making use of a result in the research by Finance Professor G. William Schwert.[11] He found that the presence of multiple bidders for control of publicly held companies on average led to increased premiums of 12.2% compared to takeovers without competitive bidding. Based on the regression in Table 4 of his article, I assume a typical deal configuration that would apply to a privately-held firm.[12] The premium without an auction was 21.5%. Adding 12.2%, the premium with an auction was 33.7%. To calculate the discount for lack of competition, we go in the other direction, i.e., 12.2% divided by one plus 33.7% = 0.122/1.337 = 9.1%, or approximately 9%. This is a useful benchmark for the second component of DLOM.

Components 3A and 3B: Buyers' and Sellers' Transactions Costs

The next two components of DLOM are the present value of an infinite continuum of transactions costs differentials for both the buyer and the seller. When using either a Discounted Cash Flow model or a Public Guideline Company method to value a business, we are beginning with the level of value known as a marketable minority interest value in the Traditional Levels of Value Chart and the (well-treated) public minority value in the 3 x 2 chart.

It is generally more expensive to buy and to sell interests in private firms than interests in public firms. For example, the brokerage fee for stocks in most public firms is 1% to 2% of the transaction price, while busi-

ness brokers usually charge 10% of the transaction price to sell private firms. Additionally, sales of interests in private firms usually entail differential legal and accounting fees, since there is no SEC scrutiny of the firms.

Transactions costs are different than the first two components (#1 Delay to Sale and #2 Monopsony Power of the Buyer). Components #1 and #2 can be passed on intact from one buyer to another, ad infinitum. As a simple example, suppose the business is worth $1 million today (time zero) and that it will be worth $1.5 million at the end of Year 10 and $2 million at the end of Year 20, etc. Suppose the Delay-to-Sale component were 5% of the value of the firm. For an immediate hypothetical sale of the business, it would be 5% of $1 million, or $50,000. Thus, the first buyer would pay $50,000 less for the business than he or she otherwise would because of Component 1. Now let's fast-forward 10 years. The first buyer now becomes the second seller, and his or her buyer is willing to pay $1.5 million for the business, less 5% x $1.5 million, or $75,000. So, Buyer #1 won the first time around by withholding $50,000, but he or she lost the second time to the tune of $75,000 (if the business would not have grown in value, then it would have been $50,000, for a net wash). In essence, the costs represented by Components 1 and 2 are passed from one owner to another and grow only with the growth in value of the business. They do not "leave the system."

Transactions costs are different than the first two components of DLOM. For Components 3A and 3B, we need to explicitly calculate the present value of the occurrence of transactions costs every time the interest sells. The reason is that, unlike the first two components, transactions costs are actually out-of-pocket costs that "leave the system." They are paid to attorneys, accountants, appraisers, and investment bankers or business brokers. Additionally, both the buyer and the seller must spend significant time on the project to make it happen, and they often have to spend time on failed acquisitions before being successful.

We need to distinguish between the buyer's transactions costs and the seller's costs. The reason for this is that the buyer's transactions costs are always relevant, whereas the seller's transactions costs for the immediate transaction reduce the net proceeds to the seller, but they do not reduce FMV. However, before the buyers are willing to buy, they should be saying, "It's true, I don't care about the sellers' costs. That's their problem. However, ten years or so down the road when it's my turn to be the seller, I do care about that." To the extent that sellers' costs exceed the brokerage cost of selling publicly-traded stock, in ten years my buyer will pay me less because of those costs, and therefore I must pay my sellers less because of my costs as a seller in Year 10. Additionally, the process goes on forever, because in Year 20, my buyer becomes a seller and faces the same problem." Thus, we need to quantify the present value of periodic buyer's transactions costs through an infinity of time beginning with the immediate sale and sellers' transactions costs that begin with the second sale of the business. With the following two formulas, we can adjust the sellers' and buyers' transactions costs to present value and calculate the resulting discount with the following formulas:

Formula for NPV of Buyers' Costs

$$D_{3A} = 1 - \frac{(1-z)(1-x^j)}{1-(1-z)x^j}, \text{ where } x = \frac{1+g}{1+r}$$

Formula for NPV of Sellers' Costs

$$D_{3B} = 1 - \frac{1-x^j}{1-(1-z)x^j}$$

In the above equations, D is the discount for transactions costs, g is the growth rate of the business, r is the discount rate of the business, j is the average number of years between transactions, and $g < r$, $0 < x < 1$. The derivation of these two equations appears in the Mathematical Appendix to Chapter 7 of QBV, pages 286-287. Analysis of partial derivatives in the Mathematical Appendix shows that the discount, i.e., DLOM, always increases with increases in growth (g) and transactions costs (z) and always decreases with increases in the discount rate (r) and the average number of years between sales (j). The converse is true as well. Decreases in the independent variables have the opposite effect of increases on DLOM.

Final Calculations of DLOM

Table V-1A is the first page of a standard three-page table to calculate DLOM in my reports. The essential calculations all appear on the first page, and the second and third pages, which I have not shown in this article, merely contain the detail supporting the calculations on page one. Let's follow through the calculations to understand how the model works.

Cell B9 shows the 3.2% discount for the Delay-to-Sale component, i.e., Component 1. We repeat that amount in C9, as there is no present value calculation necessary. In column D, we show the Remaining Value after deducting the discount due to each of the four components. For Component 1, the Remaining Value is 1 – 3.2% = 96.8% (1 – C9 = D9). For Component 2, it is 1 – 9% = 91% (D10). For Buyers' Transactions Costs, we apply the formula for Component 3A above to the pure discount, i.e., the discount at each round of selling the business, of 2.4%. The assumptions for the discount rate and the growth rate appears in D18 and D19, and their calculations and explanations would normally appear on the second and third pages of the table and on other tables in the report.

The present value of the 2.4% (B11) discount for each seller's transactions cost occurring every 10 years (where 10 is our assumption for j in the formulas on the previous page) forever is 5.7% (C11). Cell C11 contains the formula for Component 3A, so all one must do to use the ECM is to calculate or estimate B9 through B12 and D18 and D19 in Table V-1A. All the rest of the calculations are done automatically by the spreadsheet.

Similarly, the present value of the 5.1% (B12) discount for each buyer's transactions cost occurring every 10 years (where 10 is our assumption for j in the formulas on the previous page) forever is 6.7% (C12). The remaining values after subtracting C11 and C12 from one are 94.3% (D11) and 93.3% (D12), respectively.

We then calculate the remaining value in the firm by multiplying all four remaining values of the four components, i.e., 96.8% x 91.0% x 94.3% x 93.3% = 77.6% (D9 x D10 x D11 x D12 = D13). DLOM is equal to one minus the remaining value in the firm, or 1 – 77.6% = 22.4% (1 – D13 = D14). Conceptually, that is all there is to the ECM. The rest is development of the detail and documentation supporting these calculations.

This ends our section on explaining the ECM. In the next section, I provide an empirical test of the ECM versus the QMDM.

3. Empirical Test of the Two Models

The process of empirically testing QMDM vs. ECM will involve the following steps:

- Demonstrating the mathematics of the QMDM's result
- Discussing whether or not it is predictive
- Test Mercer's result in explaining the Columbia Financial Advisors, Inc. (CFAI) Study results and comparing them to the ECM calculation of the same

The Mathematics Underlying the QMDM Calculation of the Holding Period Premium (HPP)

Mercer uses a spreadsheet to back into a 30.5% implied discount rate—required holding period return, R_{HP}, in QMDM terminology—for the Management Planning, Inc. (MPI) data. It is more instructive to solve for it analytically, which we do in equations [1] – [3].

We begin with an investment of $1.00 at time zero. It grows at the marketable minority rate of return (R_{mm}) of 15% for 2.5 years (we will explain the holding period later) to $1.15^{2.5}$ = $1.42 (there is some apparent, but not real rounding error). The investor pays, on average, one dollar, less the 27.1%[13] restricted stock discount, or $1.00 – $.271 = $.729. Thus an investment of $.729 grows to $1.42 in 2.5 years. We state that growth in Equation [1]:

$$[1] \quad \$.729(1+R_{HP})^{2.5} = \$1.42$$

Dividing through by $.729, we get:

$$[2] \quad (1+R_{HP})^{2.5} = \frac{\$1.42}{\$.729} = 1.945$$

Raising both sides of [2] to the 0.4 power, we come to:

$$[3] \quad 1+R_{HP} = 1.945^{0.4} = 1.305$$

Subtracting one from both sides of equation [3] leads to the solution of the 30.5% holding period return. From there, Mr. Mercer subtracted the marketable minority return (R_{mm}) of 15% to calculate the *HPP* of 15.5%.

Predictive vs. "Post-Dictive"

Mr. Mercer stated: "If we input an *HPP* of 15.5% into Abrams' calculations, it should be obvious that a discount of 27.1% will be achieved. The QMDM is predictive of restricted stock discounts, on average, when appropriate inputs are used."

Mr. Mercer made the mistake of assuming that which he was trying to prove. He "backed into" the 15.5% *HPP* that produced a 27.1% discount, and then he claimed that the "resulting" 27.1% discount proves the accuracy of the QMDM, since it produced a 27.1%

restricted stock discount. That is not predicting the discount. It is, to coin a phrase, "post-dicting" the discount.

Table 1: An Empirical Test of Predictive Ability of the Two Models

There is a way to test both the QMDM and the ECM for their predictive abilities. Since our respective books published, Kathryn Aschwald of Columbia Financial Advisors, Inc. (CFAI) published the overall results of her firm's restricted stock study in which the Section 144 minimum holding period was reduced from two years to one year.[14] That means non-affiliates of the company can begin selling their stock after one year according to the SEC's dribbling out rules and complete selling all their stock by the end of two years. The mean time to sell in the MPI study was 2.54 years—almost exactly halfway through the year.[15] We round to 2.5 years.

As we have no knowledge of the details of the CFAI study, we make the assumption that its population had similar characteristics to the MPI study, with the exception of the holding period.[16] We recalculate the QMDM and the ECM restricted stock discounts in Table 1.

Section 1: Calculating the QMDM Restricted Stock Discount

In Section 1, rows 7 – 9, we begin with calculating the QMDM restricted stock discount for the 2.5-year holding period. We use the marketable minority interest rate of return, R_{mm}, of 15% (cell B7) and the holding period rate of return, R_{HP}, of 30.5% (B8) from Mercer's article. For every $1.00 of beginning value, the value of the enterprise should be expected to grow to $1.15^{2.5}$ = $1.418 (C7). We discount that by $1.305^{2.5}$ = 1.945 (C8). The QMDM discount is equal to 1 – (1.418/1.945) = 27.1%. This duplicates the calculation earlier in the article.[17]

In rows 12 – 14, we redo the prior calculation using an average 1.5-year holding period instead of a 2.5-year holding period, as the latter has declined by exactly one year with the change in Rule 144. Using the same formula, the QMDM forecast restricted stock discount for a 1.5 year holding period is 17.3% (C14).

Section 2: Calculating the ECM Restricted Stock Discount & Comparison to QMDM

We calculate the ECM restricted stock discount in cells B20 to B25, beginning with the MPI study restricted discount of 27.1% (B20). In B22, we show the decrease of the average holding period of one year. In B23, we insert the regression coefficient of 0.137 for the average years to sell variable.[18] Multiplying B22 x B23 = – 13.7% (B24). Adding that to B20 leads to the ECM regression forecast discount of 13.4% (B25). In C25, we repeat the QMDM forecast discount of 17.3% from C14.

In rows 26 – 29, we calculate the forecast errors and compare them. The CFAI average restricted stock discount was 13.0% (row 26). Subtracting row 26 from row 25 leads to our absolute forecast errors of 0.4% (B27) for ECM and 4.3% (C27) for QMDM. Dividing row 27 by row 26 produces the percentage forecast errors of 3.1% (B28) for ECM and 32.9% (C28) for QMDM. Dividing the QMDM error by the ECM error (C27/B27) shows that the QMDM error is 10.7, or almost 11 times the size of the ECM forecast error.[19]

Thus, the regression equation in the Economic Components Model far outperformed the QMDM in its ability to predict the CFAI results. It is my claim that the disparity in model performance will be far greater for the much longer holding periods in business valuation for a variety of reasons that I will discuss later in the article.

This concludes our empirical test of the two models. In the next section, we examine inconsistencies in Mercer's use of the QMDM.

4. Inconsistencies in the QMDM

In my view there is an inconsistency in Mercer's logic. It is a paradox that his discount rates (holding period returns) for the Management Planning, Inc. (MPI) study firms and his Chapter 10 example firms are reversed. The former should be low and the latter high, not the other way around. Mercer's attempt to explain away this paradox suffers from its own inconsistencies.

He says that if one assumes growth rates for private firms are lower than the marketable minority interest rate of return, i.e., $G_v < R_{mm}$, that may justify using a lower holding period premium—and hence, discount rate—for private firms compared to restricted stocks. We will explore these claims in detail.

The Discount Rates (Required Holding Period Returns) are Reversed

Mr. Mercer has not made a satisfactory explanation as to why the average discount rate for the privately held firms in the examples in Chapter 10 of his book is 20%, while the discount rate for the MPI firms is 30.5%. Let's review the differences of the two data sets.

The MPI firms were all publicly traded, professionally managed firms, with an average market capitalization of $78 million, and a known average 2.5-year restriction before complete marketability. The holding periods were small and certain, compared to the Chapter 10 examples, where the holding periods were generally long and uncertain. Marketability at the end of the holding period for the MPI firms was guaranteed, while marketability of the example firms was uncertain. Mercer has not adequately explained why the former should have holding period premiums that are 3 to 10 times larger than the latter.

Mercer's backing into the 30.5% holding period return results is an *ex-post* return, not an *ex-ante* return—which he did point out on page 276 of my book. However, after making that point, he appears to have ignored its implications and forgotten it. An *ex-post* return is not predictive, and it cannot be used when its underpinnings are so contrary to financial logic as they are here.

Mercer's Explanation for the Inconsistency

His explanation for that inconsistency appears in footnote 14 of his article, where he states that the appraiser's judgment may dictate that the expected growth rate in value, G_v, may be considerably lower than the marketable minority rate of return, R_{mm}. He then states, "In such cases (e.g., as in the examples provided in Chapter 10 of *Quantifying Marketability Discounts*), there is no need to 'charge' the required holding period return for uncertainties related to achieving reinvestment at the enterprise discount rate. As a result, the holding period premia (*HPP*) used by some [a]ppraisers for expected holding periods in private company valuations may be lower than those implied by the restricted stock studies." This explanation would not survive any reasonable sanity check.

Flaw in the Explanation

Let's review the concept of risk by thinking of two otherwise identical firms—one publicly held and one privately held. In finance, we think of risk as the probability distribution around our estimate of expected cash flows. In this context, there are two components of risk. The first one is the inherent business risk of being in their particular industry and market. That would be identical for the two firms and their shareholders. The second component of risk is the overlay of the risk Mercer pointed out, i.e., of being an exploited shareholder in a private firm. That increases the risk of being a private shareholder vis-à-vis a public shareholder. While abuses exist in public firms, it happens far less frequently, and there are greater remedies against this, such as class action lawsuits. The logical result is that the required holding period premium—and hence, holding period return (discount rate)—for private firms should be higher than restricted stocks, not lower.

Thus, Mercer has no logical explanation that I can perceive for the higher discount rate for the public firms with restricted stock than the private firms in his Chapter 10.

Consistent QMDM Results with a 30.5% Discount Rate

If Mercer is correct that restricted stock of publicly-traded firms with average market capitalization of $78 million and a known 2.5 year restriction before complete marketability should have discount rates of 30.5% and that the unrestricted stock has an average discount rate of around 15%, that implies that a 2.5 year period of restriction causes an average 16% (rounded) QMDM premium—well and good.

Then, imposing logical consistency, I would hazard a reasonable guess that privately-held firms with expected holding periods of 8–10 years and great uncertainty as to their length and subsequent marketability should have QMDM premiums at least 10% to 20% higher, leading to a holding period return of 40%–50% (rounded). Assuming $G_v = R_{mm}$, a 10-year holding period, and discount rates of 40% and 50%, the QMDM DLOMs are 86% (Table 2, C12) and 93% (D12), respectively, for an average of 90%.[20] Using a 50% discount rate and a 12-year holding period, the resulting DLOM is 96% (E12)—and we haven't calculated the discount for lack of control (DLOC) yet! Thus, if we impose rational consistency on the HPPs, then the QMDM calculation of DLOM for private firms produces extreme results.

For comparison, I included a QMDM calculation of a 2% DLOM in column B, based on a 1% *HPP*, which is a more appropriate *HPP* if Mercer's *s*mall *HPP*s in Chapter 10 are correct. The point is that as long as one is consistent in *HPP*s between the restricted stocks and private firms, the QMDM produces extreme results for either very short holding periods or very long holding periods. That is the major flaw of the model.

Other shortcomings of the QMDM vis-à-vis the Economic Components Model are its lack of empirical data and inability to accurately quantify the effects of thin markets and transactions cost differentials be-

tween direct ownership of the underlying assets and an ownership interest in the firm.

5. Conclusion

It should be clear from this article that criticisms of the QMDM in Chapter 7 of my book are well founded. Mercer's attempted reconciliation does not work. Mercer's only apparent defense is to bifurcate the model and use very high discount rates (R_{HP}) for publicly traded firms and low discount rates for private firms.

Meanwhile, we have seen in Table 1 that the regression equation in the ECM outperformed the QMDM by almost 11 times in forecasting the results in the CFAI Study, even when we allow the QMDM to "cheat" by using the ex-post perfect solution from the MPI Study. When we hold the QMDM to rational consistency with the examples in Mercer's Chapter 10, then the QMDM error is over 27 times larger.[21] Also, there are substantial theoretical reasons why this gap should widen considerably with ordinary business valuation, with their longer holding periods.

As indicated at the outset of this article, my intention is to continue dialogue on these and other issues confronting our profession. We have come a long way in a relatively short time. Chris Mercer, among others, has contributed to that dialogue. In fact, we both developed our DLOM models at approximately the same time in 1994, unbeknownst to the other. Ours were the first two quantitative models to calculate DLOM, and the QMDM is certainly a substantial improvement over the pure guesswork that preceded it. Let the dialogue continue.

In fact, very infrequently, I use the QMDM as a benchmark DLOM calculation. I am most tempted to do this when the holding period is very long and there is no market. For example, if I were valuing a fractional interest in a house, if the interest is not entitled to possession or income and would only achieve liquidity upon the sale, which is not expected for over 20 years, I might be tempted to include a QMDM calculation.

Endnotes

1. The author expresses his gratitude to R.K. Hiatt and Eric Nath for their helpful comments.
2. Mercer, Z. Christopher "The QMDM and Estimating Required Rates of Return for Restricted Stocks of Public Companies", *Business Valuation Review*, Vol. 20, No. 2, June 2001, p. 5.
3. See Chapter 7 of my book, *Quantitative Business Valuation* for the latest version, and "Discount for Lack of Marketability: A Theoretical Model," *Business Valuation Review*, September 1994, for the earlier version.
4. Einhorn, Hillel J. and Robin M. Hogarth, 1987, "Decision Making under Ambiguity." In *Rational Choice*, ed. Robin M. Hogarth and Melvin W. Reder: Chicago: University of Chicago Press.
5. These comments apply equally to partners in Limited Partnerships, Tenants-in-Common, and other ownership forms.
6. The Strategic Value level is a relatively new addition to the Traditional Chart, which began its existence with the lower three levels only.
7. "Should 'Marketability Discounts' be Applied to Controlling Interests of Private Companies?" *Business Valuation Review*, June 1994.
8. N.B. This chart is Bolotsky's simplification of his more complicated system. See his article, "Adjustments for Differences in Ownership Rights, Liquidity, Information Access, and Information Reliability: An Assessment of 'Prevailing Wisdom' versus the 'Nath Hypothesis'," *Business Valuation Review*, September 1991.
9. Also, public ownership affords remedies such as class action suits that private minority interests could only wish for.
10. Chaffe, David B. H., "Option Pricing as a Proxy for Discount for Lack of Marketability in Private Company Valuations—A Working Paper," *Business Valuation Review*, December 1993.
11. Schwert, G. William, 1996. "Markup Pricing in Mergers and Acquisitions." *Journal of Financial Economics 41*. 153-192.
12. We assume a successful purchase, a tender offer, and a cash deal.
13. QBV, page 238, cell C60
14. *Business Valuation Update*, May 2001.
15. These are the 53 transactions reported in *Quantitative Business Valuation*, page 238, cell I60.
16. There is another likely difference in the two populations. As shown in *Quantitative Business Valuation*, p. 128-129, the standard deviation of stock market returns as a function of company size has declined exponentially over the life of the New York Stock Exchange. The transaction dates in the MPI database range from 1980-1996, a span of 16 years, while the CFAI study had to begin after April 27, 1997 and was published in May 2001, for a maximum span of four years, with an average transaction date approximately 10 years after the MPI study. Thus, the standard deviations of returns are very unlikely to be similar, and the interest rates prevailing during the two studies are likely to be different. This renders use of Black-

Scholes less appropriate as a means to reconcile the differences in the results of the two studies.

17. An alternative form of this calculation is Discount = 1 – x^n, where $x = (1+R_{mm})/(1+R_{HP})$ and n = Holding Period. This is equivalent to discounting the $1.418 at 30.5% for 2.5 years, which leads to a value of $.729 and a discount of 27.1%.
18. *Quantitative Business Valuation*, page 240, cell B54.
19. C28/B28 leads to the same result.
20. If $G_v < R_{mm}$, that would increase DLOM further. The QMDM DLOM equals 1-x^n, where $x = (1+g)/(1+r)$, with $g = R_{mm}$ (or G_v, as appropriate), and $r = R_{HP}$. The partial derivatives of the discount are as follows:

$$\frac{\partial D}{\partial g} = -nx^{n-1}\frac{1}{1+r} < 0$$

$$\frac{\partial D}{\partial r} = -nx^{n-1}\frac{-(1+g)}{(1+r)^2} > 0$$

Thus, the QMDM DLOM is negatively related to changes in g and positively related to changes in r.

21. We begin with the CFAI result of 13.0% and subtract my 2% (actually 2.14%) calculation of the QMDM discount from Table 2, B12 to arrive at an error of 10.86%. We then divide that by the ECM error of 0.4% to arrive at a ratio of over 27.2 times the error.

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Table V-1A
Calculation of DLOM for 54% LP Interest

Section 1: Calculation of the Discount For Lack of Marketability

Component	Pure Discount = z [1]	PV of Perpetual Discount [2]	= 1- Col. [C] Remaining Value	
1	3.2%	3.2%	96.8%	Delay To Sale-1 Yr (Table V-1B, B16)
2	9.0%	9.0%	91.0%	Buyer's Monopsony Power-Thin Markets
3A	2.4%	5.7%	94.3%	Transactions Costs-Buyers [3]
3B	5.1%	6.7%	93.3%	Transactions Costs-Sellers [4]
Percent Remaining			77.6%	Total % Remaining = Components 1 x 2 x 3A x 3B
Final Discount			**22.4%**	Discount = 1 - Total % Remaining

Section 2: Assumptions and Intermediate Calculations:

Discount Rate = r (D71) [5]	**11.8%**
Constant Growth Rate = g (D74) [5]	**5.9%**
Intermediate Calculation: x = (1+g) / (1+r)	**0.9477**
Avg # Years Between Sales = j	**10**

[1] Pure Discounts: For Component #1, Table V-1B, cell B16; For Component #2, 9% per Schwert article. For Components #3A and #3B, see notes [3] and [4] below.

[2] Formula For Sellers' Discount: 1-(1-x^j)/((1-(1-z)*x^j)), per equation [7-9], used for Component #3B.
Formula For Buyers' Discount: 1-(1-z)*(1-x^j)/((1-(1-z)*x^j)), per equation [7-9a], used for Component #3A.
Components #1 and #2 simply transfer the pure discount.

[3] We calculate incremental costs for the buyer, who would have to perform due diligence on the other partners in addition to due diligence on the assets themselves, in the following manner. For the 54% LP interest, we assume the buyer would spend 20 hours at a value of his/her time of $300 per hour, or $6,000 of incremental due diligence costs. We also assume the buyer would incur legal fees of $10,000 and accounting costs of $5,000. We divide the $21,000 in due diligence costs by the Net Asset Value times 54%, the size of the LP interest, or $21,000/[($1,588,572 * 54%] = 2.4% (rounded).

Table 1
Predictive Power of QMDM vs. ECM

Section 1: Calculation of QMDM Restricted Stock Discount

	Rate of Return = R	n = Holding Per = 2.5 Yrs	Value at n = $(1+R)^{2.5}$	
	R_{mm}	15.0%	$1.418	**Value**
	R_{HP}	30.5%	1.945	**Discount Factor**
	DLOM	= $1-[(1+R_{mm})/(1+R_{HP})]^{2.5}$	**27.1%**	**1-(Value/Disc Factor)**

	Rate of Return = R	n = Holding Per = 1.5 Yrs	Value at n = $(1+R)^{1.5}$	
	R_{mm}	15.0%	$1.233	**Value**
	R_{HP}	30.5%	1.491	**Discount Factor**
	DLOM	= $1-[(1+R_{mm})/(1+R_{HP})]^{1.5}$	**17.3%**	**1-(Value/Disc Factor)**

Section 2: Calculation of ECM Restricted Stock Discount & Comparison of Errors in Both Models

	ECM	QMDM
Avg Restricted Stock Discount-MPI	27.1%	27.1%
Less:		
Decline of Section 144 Holding Period in Years	-1.0	
Regression Coefficient-Yrs 2 Sell [1]	0.137	
Forecast Decline in Restricted Stock Discount	**-13.7%**	
Forecast Restricted Stock Discount [2]	**13.4%**	**17.3%**
Avg Restricted Stock Disc-CFAI [3]	13.0%	13.0%
Absolute Forecast Error (Row 25 - Row 26)	**0.4%**	**4.3%**
Percentage Forecast Error (Row 27 / Row 26)	**3.1%**	**32.9%**
QMDM Error/ECM Error (C27 /B27)	**10.7**	

[1] *Quantitative Business Valuation: A Mathematical Approach for Today's Professionals (QBV)* , page 240, cell B54. To reconcile between the MPI and the CFAI studies, we are using the averages of the studies. Thus we use Regression 2 (page 240) rather than Regression 1 (pp. 238-239) in *QBV* , as we do not have the average price stability for the CFAI study.

[2] The ECM forecast is as calculated in B20 to B24. The QMDM forecast is from C14.

[3] CFAI is Columbia Financial Advisors, Inc. Reported in *Business Valuation Update* , May 2000.

	A	B	C	D	E
1	**Table 2**				
2	**QMDM DLOM Calculations**				
3					
4	Scenario	1	2	3	4
5	g = Growth Rate	15%	15%	15%	15%
6	r = Discount Rate	**16%**	**40%**	**50%**	**50%**
7	n = Number of Years (Holding Period)	**2.5**	10	10	**12**
8	x = (1+g)/(1+r)	0.9914	0.8214	0.7667	0.7667
9	$(1+g)^n$ = Value of Investment in 10 Years	$1.42	$ 4.05	$4.05	$5.35
10	Divide by $(1+r)^n$ to Discount to Present Value	1.45	28.93	57.67	129.75
11	$[(1+g)/(1+r)]^n = x^n$	0.979	0.140	0.070	0.041
12	QMDM Discount = $1-x^n$	**2%**	**86%**	**93%**	**96%**